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ON STABILITY AND STABILIZATION OF MOTION WITH RESPECT TO A PART OF THE VARIABLES*

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The method of Liapunov vector function is used to obtain the conditions of stability and instability of motion of a system with respect to a part of its variables. The system is described by ordinary differential equations with continuous right-hand sides. A technique for the solution of the problem concerning the stabilization of motion with respect to a part of the variables is discussed. This technique enables one to take into account the requirements previously specified, regarding the nature of the transition processes of the system, and to solve partially the problem of minimizing the demand on resources. A discussion is also presented of a method for the solution of minimization of a functional and of a game theoretic problem on a minimax-maximin of a functional, with the aim of satisfying the prescribed requirements regarding the nature of the transition processes with respect to a part of the variables in the initial system. Mechanical examples are solved.

1. Let the following system of differential equations of perturbed motion be given:

$$x^\cdot = X(t, x)\ (X(t, 0) \equiv 0) \tag{1.1}$$
$$x = (y_1, \ldots, y_m, z_1, \ldots, z_p) = (y, z),\ m > 0,\ n = m + p,$$
$$p > 0$$

We shall consider the problem of stability of the unperturbed motion $x = 0$ relative to $y_1, \ldots, y_m$ (y-stability) /1,2/. We assume /2/ that the right-hand sides of the system (1.1) are continuous in the region

$$t \geqslant 0,\ \|y\| \leqslant H > 0,\ \ 0 \leqslant \|z\| < +\infty \tag{1.2}$$

and satisfy the conditions of uniqueness of the solution. Solutions of (1.1) are z-continuable, i.e. any solution $x(t)$ is well defined at all $t \geqslant 0$, for which $\|y(t)\| \leqslant H$. We denote by $x = x(t;\ t_0,\ x_0)$ a solution of (1.1) determined by the initial conditions $x(t_0;\ t_0,\ x_0) = x_0$.

2. In a number of cases the y-stability of motion can be investigated by using the linear or nonlinear transformations of the initial system of equations to pass to an auxilliary system the Liapunov stability of which is sufficient (sometimes even necessary) for the y-stability (**) of the initial system /3,4/.

Below we study the problem of such transformations of the initial system using the differential inequalities. To solve the problem of y-stability of motion we must obtain the two-sided estimates for the variables $y = (y_1, \ldots, y_m)$ of the initial system. Such estimates will enable us to construct a Liapunov vector function which satisfies the V.M. Matrosov conditions /5/.

Theorem 2.1. Let two vector functions $V = (V_1, \ldots, V_k)$, $W = (W_1, \ldots, W_r)$ in which $V_i = W_i = y_i\ (i = 1, \ldots, m)$, $V_j = V_j(t, x)$, $W_s = W_s(t, x) (j = m+1, \ldots, k;\ s = m+1, \ldots, r)$ and for which the following conditions hold, exist in the region (1.2):

1°. $V_j(t, 0) \equiv 0$, $W_s(t, 0) \equiv 0$, $j = m+1, \ldots, k$; $s = m+1, \ldots, r$.

2°. The derivatives $V^\cdot$ and $W^\cdot$ satisfy, by virtue of the system (1.1), the inequalities

$$V_\mu^\cdot \leqslant \varphi_\mu(t, V_1, \ldots, V_k),\ \ \mu = 1, \ldots, k \tag{2.1}$$
$$W_\vartheta^\cdot \geqslant f_\vartheta(t, W_1, \ldots, W_r),\ \vartheta = 1, \ldots, r \tag{2.2}$$

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**) Such transformation of the initial system were considered in a paper by V.I. Vorotnikov: A method of investigating the stability and stabilization of a motion with respect to a part of the variables. Dissertation for degree of Candidate of Phys. and Mathem. Sciences ,Moscow,MGU, 1979.

Here the vector functions $\varphi(t, V) = (\varphi_1, \ldots, \varphi_k)$ and $f(t, W) = (f_1, \ldots, f_r)$, $\varphi(t, 0) \equiv 0$, $f(t, 0) \equiv 0$ are well defined and continuous in the region $t \geqslant 0$, $\| V^* \| < R$ where $R = \infty$, or

$$R > \sup [\| V^*(t, x) \| : t \geqslant 0, \| y \| \leqslant H], \quad V^* = (V, W)$$

3°. None of the functions $\varphi_\mu(f_\vartheta)$ decrease in V_i, $i \neq \mu$ $(W_j, j \neq \vartheta)$ /5/.

4°. Let $\alpha = (\omega_1, \ldots, \omega_m)$, $\beta = (u_1, \ldots, u_m)$. The solution $\omega = 0$ $(u = 0)$ of the system $\omega^{\cdot} = \varphi(t, \omega)$ $(u^{\cdot} = f(t, u))$ is $\alpha(\beta)$-stable.

Proof. The condition (2.2) is equivalent to

$$-W_\vartheta \leqslant f_\vartheta^*(t, -W_1, \ldots, -W_r), \ \vartheta = 1, \ldots, r.$$

and the functions f_ϑ^* do not decrease in $-W_j$ $(j \neq \vartheta)$. Let us construct the vector function $\bar{V} = (V_1, \ldots, V_k, -W_1, \ldots, -W_r)$ and put $\bar{V}^* = \max(V_s, W_s) = \max|y_s|$, $s = 1, \ldots, m$. Since the function $\bar{V}^*$ is y-positive definite, the conditions of V.M. Matrosov theorem /5/ hold for $l = 2m$, $M = \{y = 0\}$, $M_0 = \{x = 0\}$ and this proves the theorem.

Note. If the solutions $\omega = 0$ $(u = 0)$ of the corresponding systems in condition 4° are asymptotically $\alpha(\beta)$-stable, then the motion $x = 0$ of the system (1.1) is asymptotically y-stable.

Assertion 1. When $m = 1$, $k = r$ and $\varphi(t, V) \equiv f(t, W)$, and $V = W$ then a vector function V can be constructed in the region $0 \leqslant y_1 \leqslant H$, $0 \leqslant \| z \| < +\infty$ and vector function W in the region $-H \leqslant y_1 \leqslant 0$, $0 \leqslant \| z \| < +\infty$.

Proof. By virtue of the conditions $1^{\circ}, 2^{\circ}, 3^{\circ}$ and 4° of the theorem 2.1 we can find a number $\delta(\varepsilon, t_0) > 0$ such that from $\| x_0 \| < \delta$ $(y_{10} \geqslant 0)$ follows /6/

$$y_1(t; t_0, x_0) \leqslant \alpha_1^+(t; t_0, \xi_0) < \varepsilon, \quad \xi_0 = V(t_0; t_0, x_0), \ \xi_{10} \geqslant 0$$

provided that $y_1(t; t_0, x_0) \geqslant 0$, and from $\| x_0 \| < \delta$ $(y_{10} \leqslant 0)$ follows

$$y_1(t; t_0, x_0) \geqslant \alpha_1^-(t; t_0, \xi_0) > -\varepsilon, \ \xi_0 = W(t_0; t_0, x_0), \ \xi_{10} \leqslant 0$$

provided that $y_1(t; t_0, x_0) \leqslant 0$. Here α_1^+ and α_1^- are the first components of the upper $\alpha^+(t; t_0, \xi_0)$, $\alpha^+(t_0; t_0, \xi_0) = \xi_0$ and lower $\alpha^-(t; t_0, \xi_0)$, $\alpha^-(t_0; t_0, \xi_0) = \xi_0$ solution of the system

$$\xi^{\cdot} = \varphi(t, \xi) = f(t, \xi) \tag{2.3}$$

Let $t = t^*$ be the first instant of time at which $y_1(t^*, t_0, x_0) < 0$, $y_{10} \geqslant 0$. By virtue of the conditions $1^{\circ}, 3^{\circ}$ and 4° we have

$$y_1(t; t^*, x(t^*)) \geqslant \alpha_1^-(t; t^*, \xi^*), \ \xi^* = W(t^*; t^*, x(t^*))$$

for all $t \geqslant t^*$ for which $y_1(t; t_0, x_0) < 0$. But the solution $\alpha_1^-(t; t^*, \xi^*)$ of the system (2.3) can be considered as an extension of the solution $\alpha_1^-(t; t_0, \xi_0)$, $\| \xi_0 \| > \delta$, $\xi_{10} \geqslant 0$ of the same system defined for $t \in [t_0, t^*]$, on the time interval $t \geqslant t^*$ for which $y_1(t; t_0, x_0) < 0$. It follows therefore that for all $t \geqslant t_0$ we have $\| y(t; t_0, x_0) \| < \varepsilon$, provided that $\| x_0 \| < \delta$, $y_{10} \geqslant 0$. The case $\| x_0 \| < \delta$, $y_{10} \leqslant 0$ is deal with in the same manner, and this proves the assertion.

Note. Let the system (1.1) have the form

$$y^{\cdot} = Ay + Y(y, z), \ z^{\cdot} = Cy + Dz + Z(y, z),$$

where A, B and C are constant matrices and Y, Z denote the nonlinear perturbations. We assume that in region(1.2) $Y(y, z) \geqslant 0$, $A = \{a_{ij}\}$, $a_{ij} \geqslant 0$ $(i \neq j)$. In this case the Liapunov vector function can be constructed in the form $\bar{V} = (y, -\bar{W})$, while the vector function W is constructed with help of the methods of reduction to the μ-form /3,4/ using the differential inequalities.

3. Let us obtain the conditions for y-instability of the motion $x = 0$ of system (1.1). First we introduce the following definition.

Definition. The motion $x = 0$ of the system (1.1) will be called y-unstable from above (below) if numbers $\varepsilon > 0$, $t_0 \geqslant 0$ can be found such, that for any arbitrarily small $\delta(\varepsilon, t_0) > 0$ the system (1.1) has solutions $x(t; t_0, x_0)$ with initial conditions $\| x_0 \| < \delta$ satisfying, eventually, the inequality $y_i(t; t_0, x_0) > \varepsilon$ $(y_i(t; t_0, x_0) < -\varepsilon)$ for at least one $i = 1, \ldots, m$.

We introduce into the discussion two continuous scalar functions $U_1(t, x)$ and $U_2(t, x)$ and call the set of points (t, x) belonging to the region

$$t \geqslant 0, \ 0 \leqslant y_l \leqslant H, \ 0 \leqslant \| x^* \| < +\infty, \tag{3.1}$$
$$x^* = (x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n)$$

for which the inequality $U_1(t, x) > 0$ holds for at least one $l = 1, \ldots, m$, and named the region $U_1(t, x) > 0$.

Theorem 3.1. Assume that a function $U_1(t, x)$ can be found for the system (3.1), satisfying the conditions:

1) region $U_1(l) > 0$ exists for any $t \geqslant 0$ and arbitrarily small $\|x\|$;
2) function $U_1(t, x)$ is bounded in the region $U_1(l) > 0$;
3) for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any point (t, x) of the region (3.1) satisfying the condition $U_1(t, x) \geqslant \varepsilon$ we have $U_2(t, x) \geqslant \delta$ where $U_2 = U_1^{\cdot}$ is a derivative by virtue of (1.1);
4) the surface $U_1(l) = 0$ do not contains the points (t, x) for which $y_l < 0$. Then the motion $x = 0$ of the system (1.1) is y-unstable from above.

Notes. 1°. The conditions 1)—3) of the theorem 3.1 coincide with those of the Chetaev theorem /7/ on the instability (as well as with the notes of Rumiantsev in /8/) with a single difference. The region considered in /7/ was the region $t \geqslant 0, \|x\| \leqslant H$, in /8/ it was the region (1.2) and in Theorem 3.1 it is the region (3.1). Compared with /7,8/, the condition 4) is new. Conditions 1)—3) ensure that the corresponding solutions of the system (1.1) emerge from the region (3.1) after a finite time /7/. By virtue of the z-continuability of the solutions of (1.1) and of condition 4), the such solutions emerge onto the surface $y_l = H$.

2°. If the motion $x = 0$ of a linear stationary system is y-unstable, then it is y-unstable from above (from below).

3°. If instead of region (3.1) we consider the region

$$t \geqslant 0, \ -H \leqslant y_l \leqslant 0, \ 0 \leqslant \|x^*\| < +\infty$$

then the motion $x = 0$ of (1.1) under the conditions 1)—3) is y-unstable from below, provided that the surface $U_1(l) = 0$ contains no points (t, x) for which $y_l > 0$.

Theorem 3.2. Let us have at least one $l\ (1 \leqslant l \leqslant m)$ for which one of the two vector functions $V = (V_1, \ldots, V_k)$, $W = (W_1, \ldots, W_r)$ exists in the region (1.2) in which $V_1 = W_1 = y_l$, $V_j = V_j(t, x)$, $W_s = W_s(t, x)$ and for which the following conditions hold:

1°. $V_j(t, 0) \equiv 0$, $W_s(t, 0) \equiv 0$, $j \neq 1$, $s \neq 1$.

2°. The derivative $V^{\cdot}(W^{\cdot})$ satisfies in the region $t \geqslant 0, -H \leqslant y_l \leqslant V_1 \leqslant 0$ $(0 \leqslant y_l = W_1 \leqslant H)$, $0 \leqslant \|x^*\| < +\infty$, by virtue of the system (1.1), the inequality (2.1) (inequality (2.2)) where the vector function $\varphi(t, V)$ $(f(t, W))$ is defined and continuous in the region

$$\begin{gathered} t \geqslant 0, \ \|V\| < R, \ R = \infty \quad \text{or} \quad R > \sup \{\|V(t, x)\|: \ t > 0, \\ -H \leqslant y_l \leqslant 0 \\ (t \geqslant 0, \ \|W\| < \bar{R}, \ \bar{R} = \infty \quad \text{or} \quad \bar{R} > \sup \{\|W(t, x)\|: \ t \geqslant 0, \\ 0 \leqslant y_l \leqslant H) \end{gathered}$$

3°. Conditions 2° and 3° of Theorem 2.1 hold.

4°. The solution $\omega = 0$ $(u = 0)$ of the system $\omega^{\cdot} = \varphi(t, \omega)$ $(u^{\cdot} = f(t, u))$, is α_1 (β_1)- unstable from below (above). Then the motion $x = 0$ of the system (1.1) is y-unstable.

Proof. Since the solution $\omega = 0$ of the system $\omega = \varphi(t, \omega)$ is α_1-unstable from below, it follows that for any $\delta > 0$ there exist solutions $\omega(t; t_0, \omega_0) = \omega(t; t_0, V(t_0; t_0, x_0))$ with initial conditions $\|x_0\| < \delta$ $(y_{l0} \leqslant 0)$ satisfying, in the course of time, the inequality $\omega_1(t; t_0, \omega_0) < -\varepsilon$. By virtue of the condition 3° and in accordance with /6/, $y_l(t; t_0, x_0) < \omega_1^+(t; t_0, \omega_0)$, $\omega^+(t_0; t_0, \omega_0) = \omega_0$, $\omega_0 = V(t_0; t_0, x_0)$. Therefore in the course of time $y_l(t; t_0, x_0) < -\varepsilon$ and the proves the theorem.

4. Example 1. Let us consider a motion of a solid caused by the initial perturbations taking place under the action of the dissipative and acceleration forces. In this case the equation of perturbed motion has the form

$$y_1^{\cdot} = \gamma_1 y_1 + \frac{B-C}{A}, \quad z_1^{\cdot} = \gamma_2 z_1 + \frac{C-A}{B} y_1 z_2, \quad z_2^{\cdot} = \gamma_3 z_2 + \frac{A-B}{C} y_1 z_1 \tag{4.1}$$

where A, B and C are the principal moments of inertia of the body, γ_i $(i = 1, 2, 3)$ are constant numbers and $\gamma_1 < 0$, $\gamma_2 + \gamma_3 < 0$. We introduce two vector functions $V = (V_1, V_2)$, $W = (W_1, W_2)$, in which $V_1 = W_1 = y_1$, $V_2 = W_2 = z_1 z_2$. When $C < A < B$, we have the estimates

$$1) \quad V_1^{\cdot} = \gamma_1 V_1 + \frac{B-C}{A} V_2$$

$$V_2^{\cdot} = (\gamma_2 + \gamma_3) V_2 + y_1 \left[\frac{C-A}{B} z_2^2 + \frac{A-B}{C} z_1^2\right] \leqslant (\gamma_2 + \gamma_3) V_2$$

in the region $0 \leqslant y \leqslant H, 0 \leqslant \|z\| < +\infty$, and

$$2)\ W_1^{\cdot} = \gamma_1 W_1 + \frac{B-C}{A} W_2, \quad W_2^{\cdot} \geqslant (\gamma_2 + \gamma_3) W_2$$

in the region $-H \leqslant y_1 \leqslant 0$, $0 \leqslant \|z\| < +\infty$. Since the motion $\xi = 0$ of the system

$$\xi_1^{\cdot} = \gamma_1 \xi_1 + \frac{B-C}{A} \xi_2, \quad \xi_2^{\cdot} = (\gamma_2 + \gamma_3)\gamma_2$$

is asymptotically Liapunov stable, it follows in accordance with Assertion 1, that the motion $x = 0$ of the system (4.1) is asymptotically y_1- stable.

Example 2. We consider the motion of a pendulum consisting of a material point suspended from a thread the length of which varies according to an arbitrary, previously defined law $l = l(t)$, $l(t) \geqslant l_0 \geqslant 0$. We denote by θ the angle formed by the pendulum thread and the vertical. In the present case the Lagrange equation in the variables $\theta = y_1$, $\theta^{\cdot} = z_1$ has the form

$$y_1^{\cdot} = z_1, \quad z_1^{\cdot} = -\frac{g}{l(t)} \sin y_1 - 2\frac{l^{\cdot}(t)}{l(t)} z_1 \tag{4.2}$$

Let us carry out a two-sided estimation of the variables of system (4.2). We have the following inequalities:

$$1)\ y_1^{\cdot} = z_1, \quad z_1^{\cdot} \leqslant -2\frac{l^{\cdot}(t)}{l(t)} z_1$$

in the region $0 \leqslant y_1 \leqslant H$, $-\infty < z_1 < +\infty$, and

$$2)\ y_1^{\cdot} = z_1, \quad z_1^{\cdot} \geqslant -2\frac{l^{\cdot}(t)}{l(t)} z_1$$

in the region $-H \leqslant y_1 \leqslant 0$, $-\infty < z_1 < +\infty$. When the condition

$$\int_{t_0}^{t} \exp\left(\int_{t_0}^{t} -2\frac{l^{\cdot}(\tau)}{l(\tau)} d\tau\right) dt = A \int_{t_0}^{t} l(t)\, dt < \infty$$
$$A = \exp(-2)/l_0 = \text{const}$$

holds, the motion $\xi = 0$ of the system

$$\xi_1^{\cdot} = \xi_2, \quad \xi_2^{\cdot} = -2\frac{l^{\cdot}(t)}{l(t)} \xi_2(t)$$

is ξ_1- stable, therefore the motion $x = 0$ of system (4.2) is y_1-stable.

Example 3. We consider the problem of instability of the steady rotation of a solid for the Euler – Poinsot case. The equation of perturbed motion have the form

$$y_1^{\cdot} = \frac{B-C}{A} z_1 z_2, \quad z_1^{\cdot} = \frac{C-A}{B} z_2 (y_1 + p_0), \quad z_2^{\cdot} = \frac{A-B}{C} z_1 (y_1 + p_0)$$

where A, B and C are principal moments of inertia of the body and $p_0 = \text{const} > 0$. We introduce a vector function $V = (V_1, V_2)$ in which $V_1 = y_1$ and $V_2 = z_1 z_2$. When $C < A < B$ or $C > A > B$, then we have, in the region

$$y_1 \leqslant 0, \; y_1 + p_0 > 0, \; 0 \leqslant \|z\| < +\infty$$

the estimate

$$V_1^{\cdot} = \frac{B-C}{A} V_2, \quad V_2^{\cdot} = (p_0 + y_1)\left[\frac{C-A}{B} z_2^2 + \frac{A-B}{C} z_1^2\right] \leqslant 0$$

The motion $\omega = 0$ of the system

$$\omega_1^{\cdot} = \frac{B-C}{A} \omega_2, \quad \omega_2^{\cdot} = 0$$

is ω_1-unstable from below, therefore according to Theorem 3.2 the steady rotation is y_1-unstable when $C < A < B$ or when $C > A > B$. From this it follows that the steady rotation about the median axis of the inertia ellipsoid of the body is unstable relating to the projection of the angular velocity onto the axis of rotation.

5. Let us have a controlled linear system of ordinary differential equations

$$x^{\cdot} = A^* x + B^* u \tag{5.1}$$
$$x = (y_1, \ldots, y_m, z_1, \ldots, z_p) = (y, z), \; m > 0, \; n = m + p, \; p > 0$$

Here x is an n-dimensional vector characterizing the state of the system, $u=(u_1,\ldots,u_r)$ is an r-dimensional control vector, and A^*, B^* are constant $n\times n$ and $u\times z$ matrices. The system (5.1) written in the variables y and z, has the form

$$y^{\cdot}=Ay+Bz+Pu,\ z^{\cdot}=Cy+Dz+Qu \tag{5.2}$$

where A, B, C, D, P, Q are constant matrices of the corresponding sizes.

Consider the problem of stabilization of the motion $x=0$ of system (5.2) relative to $y_1,\ldots,y_m$ (y-stabilization /9,10/. We assume the controls admissible if $U=\{u\colon u=\Gamma_1 y+\Gamma_2 z\}$ where Γ_1 and Γ_2 are constant matrices of the corresponding sizes. We know /11,12/ that the vector of the controls solving the y-stabilization problem for the system (5.2) can always be constructed in the form

$$u(y,\ z)=\Gamma z+u^*(y,\ z) \tag{5.3}$$

The constant matrix Γ is determined by the transformation reducing the system (5.2) to prescribed form, and the control $u^*(y,z)$ solves the problem of stabilization over all variables for an auxilliary linear stationary system (we shall call it a system of μ-form) the dimension of which is lower than that of the initial system.

Let $\xi=(\xi_1,\ldots,\xi_N)$ denote the variables characterizing the state of the system of μ-form. We introduce the functional

$$J[u]=\int_{t_0}^{\infty}\omega(\xi[t],u^*[t])\,dt \tag{5.4}$$

where $\omega(\xi,u^*)$ is a positive-definite quadratic form of the variables ξ, u^*, $\xi[t]$ are the solutions of the system of μ-form for $u=u^*(\xi)$, $u^*[t]=u^*(\xi[t])$. Choosing the control forces u_j^*, $j=1,\ldots,r$ so that the integral (5.4) is minimized on the trajectories of the system of μ-form, we can attain the following two targets:

1°. The condition of a minimum of the integral (5.4) must ensure a sufficiently rapid decay of the motions $y_1,\ldots,y_m$ of the system (5.2) for $u=\Gamma z+u^*$, since the behavior of the variables $\xi_1,\ldots,\xi_N$ of the system of μ-form fully determines, in accordance with /11,12/, the behavior of the variables $y_1,\ldots,y_m$ of the system (5.2) for $u=\Gamma z+u^*$.

2°. The magnitude of the integral (5.4) determines satisfactorily the resources expended on forming the control $u_j^*[t]$ and hence solves partially the problem of minimizing the loss of resources in constructing the control of the form (5.3).

The problem of semioptimal y-stabilization consists of finding a control $u\in U$ such that the unperturbed motion $x=0$ of the closed system (5.2), (5.3) is asymptotically y-stable and the functional (5.4) is minimized on the trajectories of the system (5.2). Let us consider the matrices

$$K=\{(B+P\Gamma)^T,\ (D+Q\Gamma)^T(B+P\Gamma)^T,\ldots,(D+Q\Gamma)^{Tp-1}\times (B+P\Gamma)^T\}$$

$$K_1=\{L_1B^*,\ A_1L_1B^*,\ldots,A_1^{m+N-1}L_1B^*\}$$

$$L=\begin{Vmatrix}L_1\\ L_2\end{Vmatrix},\quad L_1\begin{Vmatrix}E_m & 0\\ 0 & \begin{matrix}l_{11}\ldots l_{1p}\\ l_{N1}\ldots l_{N_p}\end{matrix}\end{Vmatrix},\quad \begin{matrix}LA^*L^{-1}=\{g_{ij}\},\quad i,j=1,\ldots,n\\ A_1=\{g_{ij}\},\quad i,j=1,\ldots,m+N\end{matrix}$$

where E_m is a unit matrix of the size $m\times m$; $(l_{i1},\ldots,l_{ip})^T$, $i=1,\ldots,N$ are linearly independent column vectors -columns of the matrix K, L_2 is an arbitrary constant $(n-m-p)\times p$ matrix such that the matrix L is nondegenerate, $N=\operatorname{rank}K$ and T denotes transposition.

Theorem 5.1. Let a constant matrix Γ exist such, that $m+N=\operatorname{rank}K_1$. Then the problem of semioptimal y-stabilization has a solution for the system (5.2).

Proof. Under the conditions of the theorem a system of y-form will, according to /12/, be completely controllable. Then according to /13/ the problem of optimal stabilization up to the asymptotic Liapunov stability for the system of μ-form will, under the condition that the quality of the tansition process in a system of μ-form is determined by the functional (5.4), have a unique solution. But the behavior of the variables describing the state of the system of μ-form determines completely the behavior of the variables $y_1,\ldots,y_m$ of the closed system (5.2), (5.3), and this completes the proof of the theorem.

In the case of nonlinear systems we can also use with success the method of constructing the controls, which solve the problem of y-stabilization, in the form (5.3). We should remember the apparent infrequency of the case in which the solution controls have the form $u_j=u_j^{(1)}+u_j^{(2)}$ where the controls $u_j^{(1)}(y,z)$ reduce the initial system to some auxilliary system of μ-form (which is, in general, nonlinear) such, that the solution of the problem of optimal stabilization up to the asymptotic Liapunov stability for a system of μ-form guarantees, with help of the control $u_j^{(2)}$, the solution of the problem of semioptimal y-stabilization of the initial system by the control $u_j = u_j^{(1)} + y_j^{(2)}$. In the general case we find that the solution controls u_j have the form

$$u_j = u_j^{(1)} + f_j(y, z)\, u_j^{(2)} \tag{5.5}$$

where the controls $u_j^{(1)}$ and $u_j^{(2)}$ are defined above and $f_j(y, z)$ denote certain functions (generally speaking nonanalytic) of the variables y and z. The controls (5.5) may turn out to be nonanalytic functions of the variables determining the state of the initial system, and the problem of their physical realizability would then require additional attention. (The problem of y-stabilization of a motion within this class of controls has been studied in /12/). We note that we may use the procedures given in /4/ for constructing systems of μ-form, to construct the controls of the form (5.5).

Example 4. We consider the equations of motion of an airplane with variable aerodynamic characteristics /14/

$$\begin{gathered} x_1^{\cdot} = x_2,\ x_2^{\cdot} = a_{22}x_2 + (a_{23} + p_{23}(t))\, x_3 + b_2(t)\, u \\ x_3^{\cdot} = x_2 + (a_{33}^{\cdot} + p_{33}(t))\, x_3 \end{gathered} \tag{5.6}$$

Here x_1 is the angle of pitch, x_3 is the angle of attack, u is the deflection of the elevator, $b_2(t)$ together with the coefficients of the system represent the aerodynamic characteristics. We shall consider the problem of optimal stabilization of the unperturbed motion of the system (5.6) relative to the angle of attack. We introduce the new variable $\eta = x_2 + (a_{33} + p_{33}(t))\, x_3$, whereupon (5.6) assumes the form

$$\begin{gathered} x_1^{\cdot} = x_2,\ x_2^{\cdot} = a_{22}x_2 + (a_{23} + p_{23}(t))\, x_3 + b_2(t)\, u \\ x_3^{\cdot} = \eta,\ \eta^{\cdot} = r(t)\, x_2 + \varepsilon(t)\, x_3 + b_2(t)\, u \\ (r(t) = a_{22} + a_{33} + p_{33}(t),\ \varepsilon(t) = a_{23} + p_{23}(t) + p_{33}^{\cdot}(t) + (a_{33} + p_{33}(t))^2 \end{gathered} \tag{5.7}$$

Following (5.3) we construct the control $u(t)$ in the form

$$\begin{gathered} u(t) = -\frac{r(t)}{b_2(t)}\, x_2 + u^*(t) = \Gamma(t)\, z + u^*(t) \\ \Gamma(t) = \left(-\frac{r(t)}{b_2(t)}, 0\right), \quad z = (x_2, x_1)^T \end{gathered} \tag{5.8}$$

The following equations form the system of μ-form corresponding to the matrix $\Gamma(t)$

$$x_3^{\cdot} = \eta,\ \eta^{\cdot} = \varepsilon(t)\, x_3 + b_2(t)\, u^* \tag{5.9}$$

The problem of x_3-stabilization of the unperturbed motion of the system (5.6) is now reduced to the problem of stabilization of the motion $x_3 = \eta = 0$ of the system (5.9) up to the asymptotic Liapunov stability. Introducing the functional of the form (5.4)

$$J[u] = \int_{t_0}^{\infty} \omega(t; x_3, \eta, u^*)\, dt$$

and constructing the control u^* according to the method given in /15/, we can ensure a sufficiently rapid decay of the transition process in the closed system (5.6), (5.8), relative to the variable x_3, and solve partially the problem of minimum loss of resources in constructing the control in the form (5.8).

Example 5. We consider the problem of putting out the rotations of a heavy rigid body with a fixed point, caused by the initial perturbations. The equations of perturbed motion have the form

$$\begin{gathered} x_1^{\cdot} = \frac{B - C}{A}\, x_2x_3 + \frac{1}{A}\, mg\,(x_{30}\gamma_2 - x_{20}\gamma_3) + \frac{1}{A}\, u_1 \\ \gamma_1^{\cdot} = x_3\gamma_2 - x_2\gamma_3 \quad (123,\ ABC) \end{gathered} \tag{5.10}$$

Here A, B and C are the principal moments of inertia of the body, $x_i\ (i = 1, 2, 3)$ are the projections of the velocity of the body on the principal axes of inertia, $\gamma_i\ (i = 1, 2, 3)$ are projections of the unit vector directed along the fixed vertical axis on the principal axes of inertia, $x_{i0}\ (i = 1, 2, 3)$ are coordinates of the center of inertia in terms of the principal axes of inertia, u_1 and u_3 are controls depending on the orientation of the body and independent of its angular velocity, and u_2 is a control which may depend on the orientation of the body, as well as on its angular velocity.

Let us consider the control laws

$$u_1 = -mg\,(x_{30}\gamma_2 - x_{20}\gamma_3), \quad u_2 = -mg\,(x_{10}\gamma_3 - x_{30}\gamma_1) - \tag{5.11}$$
$$C^{-1}\left[C\,(C-A)\,x_1x_3^2 + \frac{ABC}{B-C}\,(A-B)\,x_1x_2^{\,2} + Bu^*\right] x_3\ B$$
$$u_3 = -mg\,(x_{20}\gamma_1 - x_1\gamma_2)$$

and introduce a new variable $\mu = (B-C)\,x_2x_3/A$. The following two equations will appear in the closed system (5.10), (5.11):

$$x_1^{\cdot} = \mu, \quad \mu^{\cdot} = u^* \tag{5.12}$$

and we can now replace the problem of x_1-stabilization of the unperturbed motion of the system (5.10) by a problem concerning the stabilization of the motion $x_1 = \mu = 0$ of (5.12) up to the asymptotic Liapunov stability. Introducing the functional

$$J\,[u] = \int_{t_0}^{\infty} \omega\,(x_1, \mu, u^*)\,dt$$

and solving for system (5.12) the problem of optimal stabilization of the unperturbed motion, we obtain the control laws which solve for (5.10) the problem of semioptimal x_1-stabilization. We note that the control laws (5.11) obtained map the motion of the body caused by the initial perturbations onto a plane perpendicular to the x_1-axis.

Example 6. Let us consider the equations of perturbed motion of a gyrostat /16/

$$a_1x_1^{\cdot} = x_5x_3 - x_6x_2 + u_1, \quad x_4^{\cdot} = x_5x_3 - x_6x_3 \quad (123,\ ABC) \tag{5.13}$$

where a_1, a_2, a_3 and x_1, x_2, x_3 are the moments of inertia and angular velocities of the device respectively, x_4, x_5, x_6 are linear functions of the angular velocities of the device and the flywheels, and u_1, u_2, u_3 are the controls. We consider the control laws

$$u_1 \equiv u_3 \equiv 0 \tag{5.14}$$
$$u_2 = \frac{a_2}{x_6}\left[x_5\left(\frac{x_1x_5}{a_3} - x_1x_2 - \frac{x_2x_4}{a_3}\right) + x_4\,(x_3^2 + x_2^2)\right] +$$
$$a_2\left(-x_1x_3 + \frac{x_1x_6}{a_2} - \frac{x_3x_4}{a_2}\right) + \frac{a_2}{x_6}\,u^*$$

and introduce a new variable $\mu = x_5x_3 - x_6x_2$. The following two equations will appear in the closed system (5.13), (5.14):

$$a_1x_1^{\cdot} = \mu, \quad \mu^{\cdot} = u^+ \tag{5.15}$$

and we shall be able to replace the problem of x_1-stabilization of the unperturbed motion of the system (5.13) by the problem of stabilizing the motion $x_1 = \mu = 0$ of system (5.15) up to the asymptotic Liapunov stability. Solving now the problem of optimal stabilization of the unperturbed motion for (5.15), we obtain the control laws which solve for (5.13) the problem of optimal x_1-stabilization. We note that the control laws (5.14) map the motion of the gyrostat caused by the initial perturbations, onto the plane perpendicular to the x_1-axis.

6. Considerable literature exist dealing with the problem set by A.M. Letov /16/ of minimization of a functional. The problem is that of realizing certain requirements demanded of the quality of the transition processes in the system. Let us assume that certain requirements concerning the quality of the transition processes in the variables $y_1, \ldots, y_m$ in the system (5.2) must be fulfilled. This can be done by constructing the controls of the form (5.3) and solving, for a system of equations, μ-form, the problem of minimization of a quadratic functional. We note that the problem of minimizing the loss of control resources is solved here, as in the problem of y-stabilization, only partially. By constructing the controls in the form (5.5), we can adopt an analogous approach to the nonlinear systems.

7. Let us now consider a linear system of ordinary differential equations with conflicting controls

$$x^{\cdot} = A^*x + B^*u + C^*v \tag{7.1}$$
$$x = (y_1, \ldots, y_m, z_1, \ldots, z_p) = (y, z), \quad m > 0, \quad n = m + p,$$
$$p > 0$$

Here x is an n-dimensional vector of state of the system, u and v are the control vectors of the first and second player, and A^*, B^* and C^* are constant matrices of the corresponding sizes. In the y, z-variables the system (7.1) has the form

$$y^{\cdot} = Ay + Bz + P_1u + Q_1v, \quad z^{\cdot} = Cy + Dz + P_2u + Q_2u \tag{7.2}$$

where A, B, C, D, P_1, P_2, Q_1, Q_2 are constant matrices. The linear-quadratic game for the minimax-maximin of a certain functional /17/ mirrors certain requirements concerning the behavior of the variables $x = (x_1, \ldots, x_n)$ of the initial system (7.1). Let us assume that only the requirements concerning the variables $y_1, \ldots, y_m$ of the system (7.1) must be fulfilled. This can be done by choosing the controls in the form

$$u = \Gamma^{(1)} z + u^* (t), \quad v = \Gamma^{(2)} z + v^* (t) \tag{7.3}$$

where $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are constant matrices of the corresponding size. Substituting (7.3) into (7.2) we obtain

$$\begin{aligned} y^{\cdot} &= Ay + (B + P_1\Gamma^{(1)} + Q_1\Gamma^{(2)})\, z + P_1u^* + Q_1v^* \\ z^{\cdot} &= Cy + (D + P_2\Gamma^{(2)} + Q\Gamma^{(2)})\, z + P_2u^* + Q_2v^* \end{aligned} \tag{7.4}$$

Setting now in (7.4) $u^* \equiv 0$, $v^* \equiv 0$ and constructing for the resulting system of μ-form /3/ $\xi' = G\xi$ describing fully the state of its variables $y_1, \ldots, y_m$, we obtain the following equations for the system (7.4):

$$\xi^{\cdot} = G\xi + P^*u^* + Q^*v^* \tag{7.5}$$

and the number of these equations may be lower than that of the equations in the initial system (7.1). The behavior of the variables describing the state of the system (7.5) describes fully the state of the variables $y_1, \ldots, y_m$ of the closed system (7.2), (7.3). Solving the problem for the minimax-maximin of a certain quadratic functional specified for the system (7.5), we can realize, using the known methods of its solution /17/, the specified requirements for the behavior of the variables $y_1, \ldots, y_m$ of the initial system (7.1) with conflicting controls.

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